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On the classification of quartic half-arc-transitive metacirculants

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A R T I C L E I N F O

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A B S T R A C T

A *half-arc-transitive graph* is a vertex- and edge- but not arc-transitive graph. Following Alspach and Parsons, a *metacirculant graph* is a graph admitting a transitive group generated by two automorphisms ρ and σ, where ρ is (m, n)-semiregular for some integers $m \geq 1$ and $n \geq 2$, and where σ normalizes ρ, cyclically permuting the orbits of ρ in such a way that σ^m has at least one fixed vertex. In a recent paper Marušič and the author showed that each connected quartic half-arc-transitive metacirculant belongs to one (or possibly more) of four classes of such graphs, reflecting the structure of the quotient graph relative to the semiregular automorphism ρ. One of these classes coincides with the class of the so-called tightly-attached graphs, which have already been completely classified. In this paper a complete classification of the second of these classes, that is the class of quartic half-arc-transitive metacirculants for which the quotient graph relative to the semiregular automorphism ρ is a cycle with a loop at each vertex, is given.

1. Introductory remarks

Throughout this paper graphs are assumed to be finite and, unless stated otherwise, simple, connected and undirected (but with an implicit orientation of the edges when appropriate). For group- and graph-theoretic concepts not defined here we refer the reader to [10,44] and [6], respectively.

Given a graph X, we let $V(X)$, $E(X)$, $A(X)$ and Aut$X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if its automorphism group Aut$X$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. We say that X is *half-arc-transitive* if it is vertex- and edge- but not arc-transitive. More generally, by a *half-arc-transitive* action of a subgroup $G \leq$ Aut$X$ on X we mean a vertex- and edge- but not arc-transitive action of G on X. By a classical result of Tutte [42, 7.35, p.59], a graph admitting a half-arc-transitive group action is necessarily of even valency. Since the cycles are clearly arc-transitive, the smallest admissible valency for a half-arc-transitive graph is therefore 4. In 1970 Bouwer [7] showed that the answer to the Tutte's question about existence of half-arc-transitive graphs with prescribed even valency is affirmative. He gave a construction of a $2k$-valent half-arc-transitive graph for every $k \geq 2$. The smallest graph in Bouwer's family has 54 vertices and valency 4. Some years later it was shown in [2] that the smallest half-arc-transitive graph has 27 vertices. This is the graph independently discovered by Doyle [13] and Holt [19], and is now known as the Holt graph.

The study of graphs admitting a half-arc-transitive group action, and in particular of half-arc-transitive graphs, has remained an active topic of research to this day and has been undertaken from various different aspects [2,5,8,9,14–18, 20–24,26–36,40,41,43,45–47]. (We remark that this list of references does not intend to be complete.) For the results prior to 1998 we refer also to the survey article [25].

As already noted, 4 is the smallest admissible valency for a half-arc-transitive graph, and thus special attention has rightly been given to the study of quartic half-arc-transitive graphs. Indeed, even with valency 4 the situation is so involved that

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we are currently nowhere near a complete classification of the entire class of these graphs. Nevertheless, there has been some progress. In [24] the quartic half-arc-transitive graphs were investigated via the so-called "attachment of alternating cycles" concept. It was shown that a quartic half-arc-transitive graph X possesses two parameters strongly reflecting its structural properties. First, in any of the two natural orientations of the edges of X given by the half-arc-transitive action of Aut$X$, the length of every alternating cycle, that is a cycle whose every two consecutive edges have opposite orientations, is the same. Half of this length is called the *radius* of X. Next, any two alternating cycles with nontrivial intersection meet in the same number of vertices. This number is called the *attachment number* of X. One of the possibilities is that the attachment number coincides with the radius of X. In this case we say that X is *tightly attached*. The class of quartic tightly attached half-arc-transitive graphs has been completely classified, see [24,36] for the classification of the odd and even radius graphs, respectively. It turns out that all quartic tightly attached half-arc-transitive graphs are metacirculants [24], a class of graphs that has also received a considerable amount of attention (see for example [1,3,4,11,12,20,35,37–39]). (Loosely speaking a metacirculant is a graph admitting a transitive metacyclic group $G = \langle \rho, \sigma \rangle$ where ρ is a semiregular automorphism generating a normal group in G; for a precise definition see Section 2.) On the other hand not all quartic half-arc-transitive metacirculants are tightly attached [32]. The study of these graphs is therefore the next natural step in the pursuit of a complete classification of quartic half-arc-transitive graphs. Such an investigation was initiated in [32] where it was shown that every quartic half-arc-transitive metacirculant belongs to one (or possibly more) of four classes of such graphs, reflecting the structure of the quotient graph relative to the semiregular automorphism ρ. (The four classes are described in Section 2.) It was shown that one of these classes, called Class I, coincides with the class of quartic tightly attached half-arc-transitive graphs, which, as mentioned above, has already been completely classified. It is the aim of this paper to give a complete classification of the Class II graphs, that is the class of quartic half-arc-transitive metacirculants for which the quotient graph corresponding to the semiregular automorphism ρ is a cycle with a loop at each vertex.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $m \geq 3$ and $n \geq 3$ be integers. A connected quartic graph X is a half-arc-transitive weak (m, n)-metacirculant of Class* II *if and only if $X \cong \mathcal{Y}(m, n; r, t)$, where $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ satisfy the following conditions:*

(i) $n = md_m$ *with* $d_m > 2$,
(ii) $r^2 \neq \pm 1$,
(iii) $r^m = 1$,
(iv) $m(r-1) = t(r-1) = (r-1)^2 = 0$,
(v) $\langle m \rangle = \langle t \rangle$ *in* $\mathbb{Z}_n$,
(vi) *there exists a unique* $c \in \{0, 1, \ldots, d_m - 1\}$ *such that* $t = cm$ *and* $m = ct$,
(vii) *there exists a unique* $k \in \{0, 1, \ldots, d_m - 1\}$ *such that* $kt = -km = r - 1$, *and*
(viii) *either* $m \neq 4$ *or* $t \neq 2 + 2r$.

For the definition of weak metacirculants and the graphs $\mathcal{Y}(m, n; r, t)$ see Section 2.

The proof of Theorem 1.1 is based on ideas introduced in [24,36]. The approach is to investigate the interplay of 8-cycles and 2-paths of the graph in question in order to prove or disprove its half-arc-transitivity. It is natural to ask which of the graphs $\mathcal{Y}(m, n; r, t)$ from Theorem 1.1 are pairwise isomorphic. Proposition 5.1 shows that two such graphs $\mathcal{Y}(m, n; r, t)$ and $\mathcal{Y}(m', n'; r', t')$ are isomorphic if and only if $m' = m$, $n' = n$, $t' = t$ and either $r' = r$ or $r' = r^{-1}$.

The paper is organized as follows. In Section 2 the notation is fixed and the graphs $\mathcal{Y}(m, n; r, t)$ are introduced. We also review certain results from [32] and prove some further facts about the graphs $\mathcal{Y}(m, n; r, t)$. Then, concepts linking arc- or half-arc-transitivity of $\mathcal{Y}(m, n; r, t)$ with its 8-cycles and 2-paths are introduced in Section 3. The search for possible 8-cycles is the main theme of Section 4. Finally, the proofs of Theorem 1.1 and Proposition 5.1 are given in Section 5.

2. Preliminaries

We start by introducing basic notation that will be used throughout the rest of this paper. Let X be a graph. The fact that u and v are adjacent vertices of X will be denoted by $u \sim v$; the corresponding edge will be denoted by uv. In an oriented graph the fact that the edge uv is oriented from u to v will be denoted by $u \to v$ (as well as by $v \leftarrow u$). In this case the vertex u is referred to as the *tail* and v is referred to as the *head* of the edge uv. Let U and W be disjoint subsets of $V(X)$. The subgraph of X induced by U will be denoted by $X[U]$; in short, by $[U]$, when the graph X is clear from the context. Similarly, we let $X[U, W]$ (in short $[U, W]$) denote the bipartite subgraph of X induced by the edges having one endvertex in U and the other endvertex in W.

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called (m, n)-*semiregular* if it has m orbits of length n and no other orbit. We say that a graph X is an (m, n)-*metacirculant graph* (in short an (m, n)-*metacirculant*) if there exists an (m, n)-semiregular automorphism ρ of X, together with an additional automorphism σ of X normalizing ρ, that is,

$$\sigma^{-1}\rho\sigma = \rho^r \quad \text{for some } r \in \mathbb{Z}_n^*, \tag{1}$$

and cyclically permuting the orbits of ρ in such a way that σ^m fixes a vertex of X. (Hereafter $\mathbb{Z}_n$ denotes the ring of residue classes modulo n and $\mathbb{Z}_n^*$ denotes its multiplicative group of units.) Note that this implies that σ^m fixes a vertex in every orbit of ρ. To stress the role of these two automorphisms in the definition of X we shall say that X is an (m, n)-*metacirculant relative to the ordered pair* (ρ, σ). A graph X is a *metacirculant* if it is an (m, n)-metacirculant for some m and n. This definition is equivalent with the original definition by Alspach and Parsons (see [3]). Extending this definition we say that a graph X is a *weak* (m, n)-*metacirculant* (more precisely a *weak* (m, n)-*metacirculant relative to the ordered pair* (ρ, σ)) if it has all the properties of an (m, n)-metacirculant except that we do not require that σ^m fixes a vertex of X. We say that X is a *weak metacirculant* if it is a weak (m, n)-metacirculant for some positive integers m and n.

We now present a family of quartic weak metacirculants that will play a central role in this paper.

Example 2.1. For each $m \geq 3, n \geq 3, r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$, where $r^m = 1$ and $t(r-1) = 0$, let $\mathcal{Y}(m, n; r, t)$ be the graph with vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$ and edges defined by the following adjacencies:

$$u_i^j \sim \begin{cases} u_i^{j+r^i},\ u_{i+1}^j; & i \in \mathbb{Z}_m \setminus \{m-1\},\ j \in \mathbb{Z}_n \\ u_{m-1}^{j+r^{m-1}},\ u_0^{j+t}; & i = m-1,\ j \in \mathbb{Z}_n. \end{cases} \tag{2}$$

Clearly, the permutations ρ and σ, defined by the rules

$$u_i^j\rho = u_i^{j+1}; \quad i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n \tag{3}$$

$$u_i^j\sigma = \begin{cases} u_{i+1}^{rj}; & i \in \mathbb{Z}_m \setminus \{m-1\},\ j \in \mathbb{Z}_n \\ u_0^{rj+t}; & i = m-1,\ j \in \mathbb{Z}_n, \end{cases} \tag{4}$$

are automorphisms of $\mathcal{Y}(m, n; r, t)$. Note also that ρ is (m, n)-semiregular and that $\sigma^{-1}\rho\sigma = \rho^r$. Moreover, σ cyclically permutes the orbits of ρ, and so $\mathcal{Y}(m, n; r, t)$ is a weak (m, n)-metacirculant. We remark that the Holt graph, the smallest half-arc-transitive graph (see [2,13,19]), is isomorphic to $\mathcal{Y}(3, 9; 7, 3)$. We call the edges connecting vertices of the same orbit of ρ the *inner edges* and call the edges connecting vertices from different orbits of ρ the *outer edges*. Note that each graph $X = \mathcal{Y}(m, n; r, t)$ admits an orientation consistent with the action of ρ and σ. We let $u_0^0 \to u_0^1$ and $u_0^0 \to u_1^0$ and then distribute this orientation throughout X according to the action of ρ and σ. Therefore,

$$u_i^j \to \begin{cases} u_i^{j+r^i}, u_{i+1}^j; & i \in \mathbb{Z}_m \setminus \{m-1\},\ j \in \mathbb{Z}_n \\ u_{m-1}^{j+r^{m-1}}, u_0^{j+t}; & i = m-1,\ j \in \mathbb{Z}_n. \end{cases} \tag{5}$$

Now, let X be any connected quartic half-arc-transitive weak (m, n)-metacirculant relative to the ordered pair (ρ, σ). Denote the orbits of ρ by X_i, where $i \in \mathbb{Z}_m$, in such a way that $X_i\sigma = X_{i+1}$ and let d_{inn} denote the valency of the subgraphs $[X_i]$. Furthermore, let X_ρ denote the corresponding *quotient (multi)graph relative to* ρ, whose vertex set is the set of orbits of ρ with orbits X_i and X_j connected by d edges whenever each vertex of X_i is adjacent to d vertices of X_j. As noted in Section 1, it was shown in [32] that X belongs to one (or possibly more) of the following four classes of graphs.

- *Class* I. The graph X belongs to *Class* I if $d_{inn}(X) = 0$ and each orbit X_i is connected (with a double edge) to two other orbits. In view of connectedness of X, we have that X_ρ is a "double-edge" cycle.
- *Class* II. The graph X belongs to *Class* II if $d_{inn}(X) = 2$ and each orbit X_i is connected (with a single edge) to two other orbits. In view of connectedness of X, we have that X_ρ is a cycle (with a loop at each vertex).
- *Class* III. The graph X belongs to *Class* III if $d_{inn}(X) = 0$ and each orbit X_i is connected to three other orbits, to one with a double edge and to two with a single edge. Clearly, m must be even in this case and an orbit X_i is connected to the orbit $X_{i+\frac{m}{2}}$ with a double edge. In short, X_ρ is a connected circulant with double edges connecting antipodal vertices.
- *Class* IV. The graph X belongs to *Class* IV if $d_{inn}(X) = 0$ and each orbit X_i is connected (with a single edge) to four other orbits. In short, X_ρ is a connected circulant of valency 4 and is a simple graph.

The classes I and II were extensively studied in [32]. It was shown that Class I coincides with the class of connected quartic tightly attached half-arc-transitive graphs. As for Class II, the following result can be extracted from [32, Theorem 5.1].

Proposition 2.2. *Let X be a connected quartic half-arc-transitive weak (m, n)-metacirculant of Class* II. *Then there exist $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ such that $X \cong \mathcal{Y}(m, n; r, t)$, where parameters r and t satisfy the following conditions:*

(i) $n = md_m$ *with* $d_m > 2$,
(ii) $r^2 \neq \pm 1$,
(iii) $r^m = 1$,
(iv) $m(r-1) = t(r-1) = (r-1)^2 = 0$,
(v) $\langle m \rangle = \langle t \rangle$ *in* $\mathbb{Z}_n$,
(vi) *there exists a unique* $c \in \{0, 1, \ldots, d_m - 1\}$ *such that* $t = cm$ *and* $m = ct$, *and*
(vii) *there exists a unique* $k \in \{0, 1, \ldots, d_m - 1\}$ *such that* $kt = -km = r - 1$.

For brevity reasons we make the following agreement for the rest of the paper. Whenever we refer to a condition from Proposition 2.2, we do this simply by writing the number of the condition. For example (iii) stands for $r^m = 1$. We now show that any graph $X = \mathcal{Y}(m, n; r, t)$ satisfying all the conditions (i)–(vii) admits a half-arc-transitive group of automorphisms.

Proposition 2.3. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2 and let $\rho, \sigma \in \mathrm{Aut}X$ be as in (3) and (4). Then there exists an automorphism $\tau \in (\mathrm{Aut}X)_{u_0^0}$ preserving the orientation of edges given in (5) and interchanging inner and outer edges of X. Moreover, the automorphism τ is unique and the group $H = \langle \rho, \sigma, \tau \rangle$ acts half-arc-transitively on X.*

Proof. Since $r \in \mathbb{Z}_n^*$ and since m divides n, we have that for every $i \in \mathbb{Z}_m$ and every $j \in \mathbb{Z}_n$ there exist unique integers $a \in \{0, 1, \ldots, d_m - 1\}$ and $b \in \{0, 1, \ldots, m-1\}$ such that $j = (am + b)r^i$ in $\mathbb{Z}_n$. (We say that the pair (a, b) *corresponds* to the pair (i, j).) Let $\tau : V(X) \to V(X)$ be the mapping whose action is defined according to the rule $u_i^j \tau = u_b^{i+at}$, where the pair (a, b) corresponds to the pair (i, j). We show that τ is an automorphism of X having all the required properties.

Let us show that τ is injective, and thus bijective. Suppose that for some $i, i' \in \mathbb{Z}_m$ and $j, j' \in \mathbb{Z}_n$ we have $u_i^j \tau = u_{i'}^{j'} \tau$, and let the pairs (a, b) and (a', b') correspond to (i, j) and (i', j'), respectively. Then $b = b'$ and $i - i' = (a' - a)t$ holds. Thus $i = i'$ by (v), and consequently also $a = a'$, which implies that $j = j'$. We now show that τ preserves adjacency of vertices.

Consider first the inner edges of X, that is the edges of the form $u_i^j u_i^{j+r^i}$, where $i \in \mathbb{Z}_m$, $j \in \mathbb{Z}_n$. Let (a, b) be the pair which corresponds to (i, j), that is $u_i^j \tau = u_b^{i+at}$. If $b = m - 1$ then $j + r^i = (a'm)r^i$ where $a' \in \{0, 1, \ldots, d_m - 1\}$ is such that $a' \equiv a + 1 \pmod{d_m}$. In this case (v) implies that $u_i^{j+r^i}\tau = u_0^{i+a't} = u_0^{i+at+t}$. If however $b \neq m-1$ then $j + r^i = (am + (b+1))r^i$, and so $u_i^{j+r^i}\tau = u_{b+1}^{i+at}$. In either case the edge $u_i^j u_i^{j+r^i}$ is mapped to an edge of X.

Consider next the edges of the form $u_i^j u_{i+1}^j$, where $i \in \mathbb{Z}_m \setminus \{m-1\}$ and $j \in \mathbb{Z}_n$. Let the pairs (a, b) and (a', b') correspond to the pairs (i, j) and $(i+1, j)$, respectively. Then $am + b = (a'm + b')r$, and so (iv) implies that $(a - a')m = b'r - b = b'(r-1) + b' - b$. In view of (vii) we have that $b' = b$, and so $u_i^j \tau = u_b^{i+at}$ and $u_{i+1}^j \tau = u_b^{i+1+a't}$. Let c be as in (vi) and let k be as in (vii). Then since $b = b'$ also implies that $a'm = am - b(r-1)$, we have that

$$a't = a'cm = c(am - b(r-1)) = at - bckt = at - bkm = at + b(r-1).$$

Since $(r-1)^2 = 0$, we have $r^b = (r - 1 + 1)^b = 1 + b(r-1)$, and so $u_{i+1}^j \tau = u_b^{i+at+r^b}$, proving that the edge $u_i^j u_{i+1}^j$ is mapped to an edge of X.

Finally, let us consider the edges of form $u_{m-1}^j u_0^{j+t}$, where $j \in \mathbb{Z}_n$. Let (a, b) be the pair corresponding to the pair $(m-1, j)$ and let c and k be as in (vi) and (vii), respectively. Then (iv) implies that

$$j + t = am + br^{m-1} + cm = (a + c)m + b(m-1)(r-1) + b = (a + c + bk)m + b.$$

Letting $a' \in \{0, 1, \ldots, d_m - 1\}$ be such that $a + c + bk \equiv a' \pmod{d_m}$, we thus have that $u_0^{j+t}\tau = u_b^{a't} = u_b^{at+m+bkt} = u_b^{at+m+b(r-1)}$. Since $b(r-1) = r^b - 1$, it is now clear that τ maps the edge $u_{m-1}^j u_0^{j+t}$ to an edge of X. Thus τ is an automorphism of X, as claimed. Note that τ fixes u_0^0.

In view of the nature of the action of τ it is clear that τ preserves the orientation of the edges given in (5) and that it interchanges the inner and the outer edges of X. That τ is unique is now also clear. Finally, since $\langle \rho, \sigma \rangle$ acts vertex-transitively on X, we also have that H acts half-arc-transitively on X, which completes the proof. ■

The following result can be extracted from the proof of [32, Theorem 5.1]. Here we only present the outline of the proof; for details see [32].

Proposition 2.4. *Let $X = \mathcal{Y}(4, n; r, t)$ where the parameters n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2. If $t = 2 + 2r$, then X is arc-transitive.*

Proof. It can be verified that $t = 2 + 2r$ implies that $t = \frac{n}{2} + 4$, $n \equiv 16 \pmod{32}$ and $r \equiv 5 \pmod 8$. Moreover, $r^2 = \frac{n}{2} + 1$, and so $1 + r + r^2 + r^3 + t = 8$. We introduce a certain mapping $\varphi : V(X) \to V(X)$, which can be seen to be an automorphism of X. The nature of the action of φ will show that it interchanges adjacent vertices of X which clearly implies that X is arc-transitive, as claimed.

Let $j \in \mathbb{Z}_n$. There exist unique integers $a \in \{0, 1, \ldots, 2n_1 - 1\}$ and $b \in \{0, 1, \ldots, 7\}$, where $n = 16n_1$, such that $j = 8a + b$. The action of φ on u_i^j, where $i \in \mathbb{Z}_m$, is then given in Table 1, and it depends on i and b.

It turns out that the mapping φ is a bijection and that it preserves adjacency of vertices. For details see [32]. Note that φ fixes u_0^0 and maps u_0^1 to u_0^{-1}. Thus $\varphi\rho$, where ρ is as in (3), interchanges adjacent vertices u_0^0 and u_0^1 of X, which implies that X is arc-transitive. ■

Table 1

The entry in b-th row and i-th column represents the image $u_i^j\varphi$ in the case when $j = 8a + b$, where $a \in \{0, 1, \ldots, 2n_1 - 1\}$ and $b \in \{0, 1, \ldots, 7\}$

$b \setminus i$	0	1	2	3
0	u_0^{-8a}	u_3^{-t-8a}	$u_3^{-r^3-t-8a}$	$u_2^{-r^3-t-8a}$
1	u_0^{-1-8a}	u_0^{-2-8a}	$u_3^{-2-t-8a}$	$u_3^{-2-r^3-t-8a}$
2	$u_3^{-1-t-8a}$	$u_2^{-1-t-8a}$	$u_2^{-1-r^2-t-8a}$	$u_1^{-1-r^2-t-8a}$
3	$u_3^{-1-r^3-t-8a}$	$u_3^{-1-2r^3-t-8a}$	$u_2^{-1-2r^3-t-8a}$	$u_2^{-1-r^2-2r^3-t-8a}$
4	$u_2^{-1-r^3-t-8a}$	$u_1^{-1-r^3-t-8a}$	$u_1^{-1-r-r^3-t-8a}$	$u_0^{-1-r-r^3-t-8a}$
5	$u_2^{-1-r^2-r^3-t-8a}$	$u_2^{-1-2r^2-r^3-t-8a}$	$u_1^{-1-2r^2-r^3-t-8a}$	$u_1^{-r^2-8(a+1)}$
6	$u_1^{-1-r^2-r^3-t-8a}$	$u_0^{-1-r^2-r^3-t-8a}$	$u_0^{-2-r^2-r^3-t-8a}$	$u_3^{-2-r^2-r^3-2t-8a}$
7	$u_1^{-8(a+1)}$	$u_1^{-r-8(a+1)}$	$u_0^{-r-8(a+1)}$	$u_0^{-1-r-8(a+1)}$

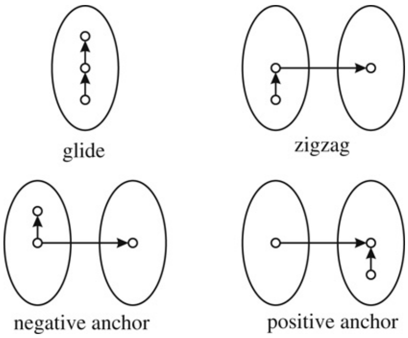

Fig. 1. Representatives of the four H-orbits of 2-paths.

3. 8-cycles versus 2-paths

For the rest of the paper let $X = \mathcal{Y}(m, n; r, t)$, where the parameters m, n, r and t satisfy conditions (i)–(vii) of Proposition 2.2, and let ρ and σ be as in (3) and (4). Furthermore, let τ be as in Proposition 2.3 and let $H = \langle \rho, \sigma, \tau \rangle$. The proof of Theorem 1.1 is based on the investigation of the interplay of H-orbits of 8-cycles of X and H-orbits of 2-paths of X. (Observe that X does possess 8-cycles - one of them is $u_0^0 u_0^1 u_1^1 u_1^{1+r} u_0^{1+r} u_0^r u_1^r u_1^0$.)

Following the theory developed in [24] and later in [36] we introduce the following notation concerning 2-paths of X. Observe first that the group H has four orbits in its natural action on the set of 2-paths of X. We call the 2-paths belonging to the H-orbit of $u_0^0 u_1^0 u_1^{-r}$ the *positive anchors*, the 2-paths belonging to the H-orbit of $u_0^1 u_0^0 u_1^0$ the *negative anchors*, the 2-paths belonging to the H-orbit of $u_0^0 u_0^1 u_0^2$ the *glides* and the 2-paths belonging to the H-orbit of $u_0^0 u_0^1 u_1^1$ the *zigzags* (see Fig. 1). These four H-orbits of 2-paths of X will be denoted by Anc^+X, Anc^-X, $\mathrm{Gli}X$ and $\mathrm{Zig}X$, respectively. At times it will be convenient not to distinguish between positive and negative anchors; in such cases we will refer to them simply as *anchors*. Observe that glides are characterized by the fact that its two corresponding edges are both inner or both outer. Note also that there is precisely one positive and precisely one negative anchor having a given vertex as its internal vertex, whereas there are precisely two glides and precisely two zigzags having a given vertex as its internal vertex. Thus we have

$$|\mathrm{Anc}^+X| = |\mathrm{Anc}^-X| = mn \quad \text{and} \quad |\mathrm{Gli}X| = |\mathrm{Zig}X| = 2mn. \tag{6}$$

The next proposition links the problem of determining whether the graph X is half-arc-transitive or not to the investigation of the action of its automorphism group on the set of 2-paths of X.

Proposition 3.1. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2. If X is arc-transitive then the automorphism group* $\mathrm{Aut}X$ *does not fix the set of glides* $\mathrm{Gli}X$.

Proof. Suppose on the contrary that X is arc-transitive and that the set $\mathrm{Gli}X$ is an orbit of $\mathrm{Aut}X$. Then there exists some $\varphi \in \mathrm{Aut}X$ fixing u_0^0 such that $u_0^1\varphi = u_0^{-1}$. Since φ preserves the set of glides of X, we must have $u_0^{-1}\varphi = u_0^1$. Moreover, since $u_0^0 u_0^1 u_0^2$ also is a glide, we have $u_0^2\varphi = u_0^{-2}$. Continuing inductively we see that φ interchanges the vertices u_0^j and u_0^{-j} for every $j \in \mathbb{Z}_n$. We have two possibilities for $u_1^0\varphi$.

Suppose first that φ fixes the vertex u_1^0. Then the fact that $u_0^0 u_1^0 u_2^0 \in \mathrm{Gli}X$ forces $u_2^0\varphi = u_2^0$. Continuing inductively we see that φ fixes all the vertices of form u_i^0, where $i \in \mathbb{Z}_m$. Moreover, since $u_{m-2}^0 u_{m-1}^0 u_0^t \in \mathrm{Gli}X$, we also have $u_0^t\varphi = u_0^t$. But $u_0^t\varphi = u_0^{-t}$, which implies that $2t = 0$, contradicting the fact that $\langle m \rangle = \langle t \rangle$ and $d_m > 2$. (Recall that $n = md_m$.)

Suppose now that $u_1^0\varphi = u_{m-1}^{-t}$. Then the fact that $u_0^0u_1^0u_1^r$ is not a glide forces $u_1^r\varphi \in \{u_{m-1}^{-t+r^{m-1}}, u_{m-1}^{-t-r^{m-1}}\}$. But $u_0^r \sim u_1^r$, and so u_0^{-r} is adjacent to one of the vertices $u_{m-1}^{-t+r^{m-1}}$ and $u_{m-1}^{-t-r^{m-1}}$. Hence, either $r = -r^{m-1}$ or $r = r^{m-1}$, and so $r^2 = \pm 1$, a contradiction. ■

We now introduce some further notation. Let W be a simple walk of length d in X. To each internal vertex v of W we assign one of the symbols a, g or z, depending on whether the corresponding 2-path of W having v as its internal vertex is an anchor, a glide or a zigzag, respectively. In this way a sequence of symbols from the set $\{a, g, z\}$ is assigned to W. If W is a cycle, then every vertex of W is internal so that the length of the obtained sequence is d. Otherwise the length of the sequence is $d-1$. We let the equivalence class of all sequences obtained from the above sequence by reflections and cyclic rotations in case when W is a cycle, and just by a reflection in case when W is a path, be the *code* of W. The code of W will be denoted by any one of its representatives.

The next observation is clear.

Proposition 3.2. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2, and let C be an 8-cycle of X containing at least one anchor. Then positive and negative anchors alternate on C.*

Let $\mathcal{C}$ be a union of H-orbits of 8-cycles of X and let P be a 2-path of X. The number of cycles of $\mathcal{C}$ containing P as a subgraph will be called the $\mathcal{C}$*-frequency* of P and will be denoted by $f(\mathcal{C}, P)$. The next result is straightforward.

Proposition 3.3. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2. Let G be a subgroup of automorphisms of X such that $H \le G \le \mathrm{Aut}X$. Let $\mathcal{C}$ be a union of G-orbits of 8-cycles of X and let P and Q be any two 2-paths of X permutable by some element of G. Then $f(\mathcal{C}, P) = f(\mathcal{C}, Q)$.*

Let $\mathcal{C}$ be a union of H-orbits of 8-cycles of X and let x be any of the symbols a^+, a^-, g and z. By Proposition 3.3 we can now define $f_x(\mathcal{C})$ to be the frequency $f(\mathcal{C}, P)$ where P is any 2-path of type x. Note that by Proposition 3.2 we have $f_{a^+}(\mathcal{C}) = f_{a^-}(\mathcal{C})$, and so we set $f_a(\mathcal{C}) = f_{a^+}(\mathcal{C}) = f_{a^-}(\mathcal{C})$. The next lemma gives an easy way of calculating these frequencies and will be used throughout the rest of the paper without special reference to it.

Lemma 3.4. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2. Let $\mathcal{C}$ be an H-orbit of 8-cycles of X and let x be any symbol from the set $\{a, g, z\}$. Let $C \in \mathcal{C}$ and suppose C contains $\varepsilon_{x,C}$ 2-paths of type x. Then*

$$f_x(\mathcal{C}) = \frac{|\mathcal{C}| \cdot \varepsilon_{x,C}}{2mn}.$$

Proof. Let P be any 2-path of type x and let $\mathcal{P}$ be its H-orbit. We let *Bip* be the bipartite graph having as the two bipartition sets $\mathcal{C}$ and $\mathcal{P}$ such that $C' \in \mathcal{C}$ is adjacent to $P' \in \mathcal{P}$ whenever C' contains P' as a subgraph. Since H acts transitively on each of the two bipartition sets, a simple counting argument together with Proposition 3.3 shows that $|\mathcal{C}| \cdot \varepsilon_{x,C} = |\mathcal{P}| \cdot f_x(\mathcal{C})$ if $x \in \{g, z\}$. By (6), the result follows. Suppose now that $x = a$. Then Proposition 3.2 implies that C contains $\frac{\varepsilon_{a,C}}{2}$ positive and $\frac{\varepsilon_{a,C}}{2}$ negative anchors. By the above argument we thus have that $|\mathcal{C}| \cdot \frac{\varepsilon_{a,C}}{2} = |\mathrm{Anc}^+X| \cdot f_{a^+}(\mathcal{C}) = |\mathrm{Anc}^+X| \cdot f_a(\mathcal{C})$, which, by (6), completes the proof. ■

Combining together Propositions 2.3, 3.1 and 3.3, we obtain the following result, which will be used throughout the rest of the paper.

Proposition 3.5. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2 and let $\mathcal{C}$ be a union of $\mathrm{Aut}X$-orbits of 8-cycles of X. If $f_g(\mathcal{C}) \neq f_a(\mathcal{C})$ and $f_g(\mathcal{C}) \neq f_z(\mathcal{C})$ then X is half-arc-transitive.*

The following proposition will be helpful in the search for all H-orbits of 8-cycles of X.

Proposition 3.6. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2. Then the parameters m, n, r^{-1} and t also satisfy conditions* (i)–(vii) *of Proposition 2.2 and $X \cong \mathcal{Y}(m, n; r^{-1}, t)$. Moreover, there exists such an isomorphism $\varphi : X \to Y$, where $Y = \mathcal{Y}(m, n; r^{-1}, t)$, that $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$, $(\mathrm{Zig}X)\varphi = \mathrm{Zig}Y$, $(\mathrm{Anc}^+X)\varphi = \mathrm{Anc}^+Y$ and $(\mathrm{Anc}^-X)\varphi = \mathrm{Anc}^-Y$.*

Proof. That the parameters m, n, r^{-1} and t satisfy the conditions (i)–(vii) of Proposition 2.2 is straightforward. We leave this to the reader.

To prove that the two graphs are isomorphic denote the vertices of $Y = \mathcal{Y}(m, n; r^{-1}, t)$ by v_i^j, $i \in \mathbb{Z}_m$, $j \in \mathbb{Z}_n$, with adjacencies as in (2), and let $\varphi : X \to Y$ be the mapping defined by the rule

$$u_i^j\varphi = \begin{cases} v_0^{-i-at}; & b = 0 \\ v_{m-b}^{-ir^b-(a+1)t}; & b \neq 0, \end{cases} \tag{7}$$

where $a \in \{0, 1, \ldots, d_m - 1\}$ and $b \in \{0, 1, \ldots, m-1\}$ are the unique integers for which $j = am + b$. (Similarly as in the proof of Proposition 2.3 we say that the pair (a, b) *corresponds* to j.) We now show that φ is an isomorphism of graphs.

We first prove that φ is injective and thus bijective. Suppose that for some $i, i' \in \mathbb{Z}_m$ and $j, j' \in \mathbb{Z}_n$ we have that $u_i^j\varphi = u_{i'}^{j'}\varphi$. Let the pairs (a, b) and (a', b') correspond to j and j', respectively. Then (7) implies that $b = b'$. If $b = 0$, then $-i - at = -i' - a't$, and so $i - i' = (a' - a)t$. By (v) we thus have that $i = i'$ and consequently $a' = a$, implying that $j = j'$, as required. If however $b \neq 0$, then $-ir^b - (a+1)t = -i'r^b - (a'+1)t$, and so (iv) implies that $-i - (a+1)t = -i' - (a'+1)t$. We can now proceed as in the case $b = 0$. Thus φ is a bijection, as claimed.

Let us now show that φ preserves adjacency of vertices. Recall that the orbits of ρ, where ρ is as in (3), are denoted by X_i. Since by (iii) we have that $r^b = (r^{-1})^{m-b}$, it is clear that the edges of $[X_i, X_{i+1}]$, where $i \in \mathbb{Z}_m \setminus \{m-1\}$, are mapped to edges of Y. Next, consider adjacent vertices u_{m-1}^j and u_0^{j+t}, and let (a, b) be the pair corresponding to j. Let c be as in (vi) and let $a' \in \{0, 1, \ldots, d_m - 1\}$ be such that $a + c \equiv a' \pmod{d_m}$. If $b = 0$, then (7) implies that $u_{m-1}^j\varphi = v_0^{-m+1-at}$ and $u_0^{j+t}\varphi = v_0^{-a't} = v_0^{-at-ct} = v_0^{-at-m}$. If however $b \neq 0$, then

$$
\begin{aligned}
u_{m-1}^j\varphi &= v_{m-b}^{-(m-1)r^b-(a+1)t} = v_{m-b}^{-m+r^b-(a+1)t} \quad \text{and}\\
u_0^{j+t}\varphi &= v_{m-b}^{-(a'+1)t} = v_{m-b}^{-(a+1)t-ct} = v_{m-b}^{-(a+1)t-m}.
\end{aligned}
$$

Thus in any case $u_{m-1}^j\varphi$ and $u_0^{j+t}\varphi$ are adjacent. Finally, let $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$, let (a, b) be the pair corresponding to j and let k be as in (vii). By (iv) we have that $r^i = (r-1+1)^i = \mathrm{i}(r-1) + 1 = -\mathrm{i}km + 1$. If $b = m-1$ then let $a' \in \{0, 1, \ldots, d_m - 1\}$ be such that $a' \equiv a - \mathrm{i}k + 1 \pmod{d_m}$ and let $b' = 0$, and if $b \neq m-1$ then let $a' \in \{0, 1, \ldots, d_m - 1\}$ be such that $a' \equiv a - \mathrm{i}k \pmod{d_m}$ and let $b' = b + 1$. Therefore $j + r^i = a'm + b'$. Depending on b we have three different possibilities to check. We check the case $0 < b < m-1$ and leave the cases $b = 0$ and $b = m-1$ to the reader. Suppose then that $0 < b < m-1$. Then $b' \neq 0$, and so (7) implies that

$$
u_i^j\varphi = v_{m-b}^{-ir^b-(a+1)t} \quad \text{and} \quad u_i^{j+r^i}\varphi = v_{m-b'}^{-ir^{b'}-(a'+1)t} = v_{m-b-1}^{-ir^{b+1}-(a+1)t+\mathrm{i}(r-1)}.
$$

Since

$$
\mathrm{i}r^{b+1} = \mathrm{i}((b+1)(r-1)+1) = \mathrm{i}(b(r-1)+1) + \mathrm{i}(r-1) = \mathrm{i}r^b + \mathrm{i}(r-1),
$$

it is now clear that $u_i^j\varphi$ and $u_i^{j+r^i}\varphi$ are adjacent. Therefore, φ is an isomorphism of graphs, and so $X \cong Y$. That φ has the required properties concerning 2-paths of X and Y is evident from the nature of its action. ■

4. The 8-cycles

In this section we determine the possible H-orbits of 8-cycles of X. To do so a thorough investigation of all the possibilities has to be undertaken. Observe that since we are searching for H-orbits of 8-cycles and since $\tau \in H$ interchanges inner and outer edges of X it suffices to consider only 8-cycles with at most four outer edges. The investigation is divided depending on the number of outer edges the 8-cycle in question contains. The following lemma gives all the possible H-orbits of 8-cycles of X together with a representative, its code, a necessary and sufficient condition for its existence and the orbit length.

Lemma 4.1. *Let $X = \mathcal{Y}(m, n; r, t)$ where the parameters m, n, r and t satisfy conditions* (i)–(vii) *of Proposition 2.2 and let H be as in Proposition 2.3. Suppose that $n \neq 16$. Then the possible H-orbits of 8-cycles of X are those given in Table 2, where each H-orbit is given by a representative, together with its code, a necessary and sufficient condition for its existence and the orbit length.*

Proof. Recall our agreement that any of (i)–(vii) stands for the corresponding item of Proposition 2.2. Note first that since $m \geq 3$, conditions (i) and (ii) imply that $n \geq 9$ when n is odd and that $n \geq 18$ when n is even. Let C be an 8-cycle of X. As already mentioned in the paragraph preceding the statement of this lemma we can assume that the number *out* of outer edges of C is at most four. It is thus clear that *out* is even unless $m = 3$ in which case out $= 3$ can occur. We distinguish four different cases depending on the number *out*.

CASE 1: out $= 0$. In this case clearly $8 \equiv 0 \pmod{n}$ must hold which is impossible in view of $n \geq 9$.

CASE 2: out $= 2$. It is clear that in this case C contains only vertices from two consecutive orbits X_i. (Recall that the sets X_i, where $i \in \mathbb{Z}_m$, denote the orbits of ρ, where ρ is as in (3).) Since we are considering H-orbits of 8-cycles and since $\sigma \in H$, we can assume that these two orbits are X_0 and X_1. Depending on the maximal number of consecutive inner edges of C we have that the H-orbit of C corresponds in a natural way to one of the conditions $5 \pm r = 0$, $1 \pm 5r = 0$, $4 \pm 2r = 0$, $2 \pm 4r = 0$ and $3 \pm 3r = 0$ (recall that $\sigma \in H$). We consider each of them separately.

• $5 \pm r = 0$. If $5 - r = 0$, then $r - 1 = 4$, and so (iv) implies that $16 \equiv 0 \pmod{n}$. But then $n = 16$, which, by assumption, does not hold. If however $5 + r = 0$, then $r - 1 = -6$, and so $36 \equiv 0 \pmod{n}$. We have several possibilities. If $n = 36$, then $r = 31$, and so combining together (i), (iv) and (vii) we find that $m = 6$ and $t = 30$. Thus $X = \mathcal{Y}(6, 36; 31, 30)$ in this case. If $n = 18$, then $r = 13$. Similar arguments as above show that then $X = \mathcal{Y}(3, 18; 13, 15)$ or $X = \mathcal{Y}(6, 18; 13, 12)$. By the

Table 2
The possible H-orbits of 8-cycles in X

Row	Representative	Code	Condition	Orbit length
1	$u_0^0u_0^1u_1^1u_1^{1+r}u_0^{1+r}u_0^ru_1^ru_1^0$	a^2zaz^2az	none	$2mn$
2	$u_0^0u_0^1u_1^1u_2^1u_2^{1+r^2}u_1^{1+r^2}u_1^{1-r+r^2}u_1^0$	$azgzazgz$	none	mn
3	$u_0^0u_0^1u_1^1u_1^{1-r}u_1^{1-2r}u_2^{1-2r}u_2^0u_1^0$	$azagazag$	none	mn
4	$u_0^0u_0^1u_0^2u_0^3u_1^3u_1^{3-r}u_1^{3-2r}u_1^{3-3r}$	ag^2zag^2z	$3(r-1)=0$	$2mn$
5	$u_0^0u_0^1u_0^2u_1^2u_1^{2-r}u_2^{2-r}u_2^{2-r-r^2}u_1^{2-r-r^2}$	a^3zgagz	$3(r-1)=0$	mn
6	$u_0^0u_0^1u_1^1u_2^1u_2^{1-r^2}u_2^{1-2r^2}u_1^{1-2r^2}u_1^{1+r-2r^2}$	a^3zgagz	$3(r-1)=0$	mn
7	$u_0^0u_0^1u_0^2u_1^2u_2^2u_2^{2-r^2}u_1^{2-r^2}u_1^{2-r-r^2}$	az^3agzg	$3(r-1)=0$	$2mn$
8	$u_0^0u_0^1u_0^2u_1^2u_2^2u_2^{2-r^2}u_2^{2-2r^2}u_1^{2-2r^2}$	$agzgagzg$	$4(r-1)=0$	mn
9	$u_0^0u_0^1u_0^2u_1^2u_1^{2+r}u_1^{2+2r}u_1^{2+3r}u_1^{2+4r}$	$a^2g^3z^2g$	$(m,n,r,t)\in A_1$	$2mn$
10	$u_0^0u_0^1u_0^2u_0^3u_0^4u_1^4u_1^{4+r}u_1^{4+2r}$	$a^2g^3z^2g$	$(m,n,r,t)\in A_2$	$2mn$
11	$u_0^0u_0^1u_0^2u_0^3u_0^4u_0^5u_1^5u_1^{5+r}$	$a^2g^4z^2$	$(m,n,r,t)\in B_1$	$2mn$
12	$u_0^0u_0^1u_1^1u_1^{1+r}u_1^{1+2r}u_1^{1+3r}u_1^{1+4r}u_1^{1+5r}$	$a^2g^4z^2$	$(m,n,r,t)\in B_2$	$2mn$
13	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_0^{-4}u_1^{-4}u_1^{-4+r}u_2^{-4+r}$	ag^3az^2g	$m=3$ and $t=4-r$	$2mn$
14	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_1^{-3}u_1^{-3-r}u_2^{-3-r}u_2^{-3-r+r^2}$	$a^3g^2az^2$	$m=3$ and $t=4-r$	$2mn$
15	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_0^{-4}u_1^{-4}u_2^{-4}u_2^{-4+r^2}$	ag^3az^2g	$m=3$ and $t=2+r$	$2mn$
16	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_1^{-3}u_1^{-3+r}u_2^{-3+r}u_2^{-3+r-r^2}$	$a^3g^2az^2$	$m=3$ and $t=2+r$	$2mn$
17	$u_0^0u_0^1u_0^2u_1^2u_1^{2+r}u_1^{2+2r}u_2^{2+2r}u_2^{2+2r-r^2}$	a^2zgz^2gz	$m=3$ and $3+t=0$	$2mn$
18	$u_0^0u_0^1u_0^2u_0^3u_0^4u_1^4u_2^4u_3^4$	g^3zg^3z	$m=4$ and $4+t=0$	mn
19	$u_0^0u_0^1u_0^2u_1^2u_2^2u_2^{2+r^2}u_2^{2+2r^2}u_3^{2+2r^2}$	$gzgzgzgz$	$m=4$ and $4+t=0$	$2n$
20	$u_0^0u_0^1u_0^2u_1^2u_1^{2+r}u_2^{2+r}u_3^{2+r}u_3^{2+r+r^3}$	gz^3gz^3	$m=4$ and $4+t=0$	mn
21	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_1^{-3}u_1^{-3-r}u_2^{-3-r}u_3^{-3-r}$	$a^3g^2ag^2$	$m=4$ and $t=3+r$	mn
22	$u_0^0u_0^{-1}u_0^{-2}u_1^{-2}u_2^{-2}u_2^{-2-r^2}u_3^{-2-r^2}u_3^{-2-r^2-r^3}$	a^5gag	$m=4$ and $t=3+r$	mn
23	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_1^{-3}u_2^{-3}u_3^{-3}u_3^{-3-r^3}$	$a^3g^2ag^2$	$m=4$ and $t=1+3r$	mn
24	$u_0^0u_0^{-1}u_0^{-2}u_1^{-2}u_1^{-2-r}u_2^{-2-r}u_2^{-2-r-r^2}u_3^{-2-r-r^2}$	a^5gag	$m=4$ and $t=1+3r$	mn
25	$u_0^0u_0^{-1}u_0^{-2}u_0^{-3}u_1^{-3}u_2^{-3}u_2^{-3-r^2}u_3^{-3-r^2}$	a^2gag^2ag	$m=4$ and $t=2+2r$	$2mn$
26	$u_0^0u_0^{-1}u_1^{-1}u_1^{-1-r}u_2^{-1-r}u_2^{-1-r-r^2}u_3^{-1-r-r^2}u_3^{-1-r-r^2-r^3}$	a^8	$m=4$ and $t=2+2r$	n
27	$u_0^0u_0^1u_0^2u_0^3u_1^3u_2^3u_2^{3+r^2}u_3^{3+r^2}$	g^2zgz^2gz	$m=4$ and $t=-2-2r$	$2mn$
28	$u_0^0u_0^1u_1^1u_1^{1+r}u_2^{1+r}u_2^{1+r+r^2}u_3^{1+r+r^2}u_3^{1+r+r^2+r^3}$	z^8	$m=4$ and $t=-2-2r$	n

Here $A_1=\{(3,18,13,15),(6,18,13,12),(3,9,4,6)\}$, $A_2=\{(3,18,7,15),(6,18,7,12),(3,9,7,6)\}$, $B_1=A_1\cup\{(6,36,31,30)\}$ and $B_2=A_2\cup\{(6,36,7,30)\}$.

remarks from the first paragraph of this proof we are left with the possibility $n=9$ in which case $X=\mathcal{Y}(3,9;4,6)$ is the Holt graph. It is clear that the 8-cycles corresponding to the condition $5+r=0$ are of code $a^2g^4z^2$ and that the length of the corresponding H-orbit is $2mn$. Thus this H-orbit corresponds to row 11 of Table 2.

• $1\pm 5r=0$. By Proposition 3.6 and the above paragraph $1-5r=0$ is impossible, since then $5-r^{-1}=0$. Moreover, the H-orbit corresponding to $1+5r=0$ exists if and only if X is one of $\mathcal{Y}(6,36;7,30)$, $\mathcal{Y}(3,18;7,15)$, $\mathcal{Y}(6,18;7,12)$ and $\mathcal{Y}(3,9;7,6)$ and corresponds to row 12 of Table 2.

• $4\pm 2r=0$. Note that $4-2r=0$ cannot hold, since then (iv) implies that $0=4r-2r^2=2$, which contradicts $n\geq 9$. Suppose now that $4+2r=0$. In this case $0=4r+2r^2=8r-2$, and so $18\equiv 0 \pmod n$. It follows that $n\in\{9,18\}$. If $n=9$, then $2(2+r)=0$ implies that $r=7$, and so (i) and (vii) imply that $X=\mathcal{Y}(3,9;7,6)$. If however $n=18$, then $2(2+r)=0$ also implies that $r=7$ (recall that $r\in\mathbb{Z}_{18}^*$), and so X is one of the graphs $\mathcal{Y}(3,18;7,15)$ and $\mathcal{Y}(6,18;7,12)$. Observe that the 8-cycles corresponding to the condition $4+2r=0$ are of code $a^2g^3z^2g$ and that the length of the corresponding H-orbit is $2mn$. Thus this H-orbit corresponds to row 10 of Table 2.

• $2\pm 4r=0$. By Proposition 3.6 and the above paragraph $2-4r=0$ is impossible. Moreover, $2+4r=0$ holds if and only if X is one of the graphs $\mathcal{Y}(3,9;4,6)$, $\mathcal{Y}(3,18;13,15)$ and $\mathcal{Y}(6,18;13,12)$ and thus the H-orbit in question corresponds to row 9 of Table 2.

• $3\pm 3r=0$. The condition $3+3r=0$ is not possible, since then $0=3r+3r^2=9r-3$, and so $12\equiv 0 \pmod n$, which is not possible. On the other hand, $3-3r=0$ is possible. It is clear that the 8-cycles corresponding to this H-orbit are of code ag^2zag^2z and that the length of the orbit is $2mn$. Thus this H-orbit corresponds to row 4 of Table 2.

CASE 3: out $=3$. As noted at the beginning of this proof, we have $m=3$ in this case. Depending on the maximal number of consecutive inner edges of C we have that the H-orbit of C corresponds in a natural way to one of the conditions $\pm 5+t=0$,

$\pm 4 \pm r + t = 0$ or $\pm 4 \pm r^2 + t = 0$, $\pm 3 \pm r \pm r^2 + t = 0$, $\pm 3 \pm 2r + t = 0$ or $\pm 3 \pm 2r^2 + t = 0$, and $\pm 2 \pm 2r \pm r^2 + t = 0$. Observe first that (vii) implies that $r - 1 \in \langle 3 \rangle$, and so $r + 1 \notin \langle 3 \rangle$. Thus, in view of (v), we immediately find that none of the conditions $\pm 5 + t = 0$, $\pm(4 + r) + t = 0$, $\pm(4 + r^2) + t = 0$, $\pm 3 \pm (r + r^2) + t = 0$, $\pm 3 \pm 2r + t = 0$, $\pm 3 \pm 2r^2 + t = 0$ and $\pm(2 - 2r) \pm r^2 + t = 0$ can hold. As for the remaining conditions we consider each of them separately.

• $\pm(4 - r) + t = 0$. If $4 - r + t = 0$ then $C_0 = u_0^0 u_0^1 u_0^2 u_0^3 u_0^4 u_1^4 u_1^{4-r} u_2^{4-r}$ is an 8-cycle. Applying τ we find that $C_1 = u_0^0 u_1^0 u_2^0 u_0^t u_1^t u_1^{r+t} u_0^{r+t} u_0^{1+r+t}$ also is an 8-cycle, and so $2 + r + t = 0$. But then $4 - r + t = 0$ implies that $-2 + 2r = 0$, and so (iv) implies that $r^2 = (r - 1 + 1)^2 = 1$, which contradicts (ii). On the other hand, $-4 + r + t = 0$ is possible. The 8-cycles corresponding to this condition are of code ag^3az^2g and the length of the corresponding H-orbit is $6n$. Thus, this H-orbit corresponds to row 13 of Table 2.

• $\pm(4 - r^2) + t = 0$. Since $r^2 = r^{-1}$, Proposition 3.6 implies that only $-4 + r^2 + t = 0$ is possible. By (iv) we have that $0 = -4 + r^2 + t = -5 + 2r + t = -2 - r + t$, and so the H-orbit in question corresponds to row 15 of Table 2.

• $\pm 3 \pm (r - r^2) + t = 0$. Note that this condition is equivalent to $\pm 3 \pm (1 - r) + t = 0$. Since we already know that $4 - r + t = 0$ and $2 + r + t = 0$ are not possible, we are left with two possibilities. It is easy to see that the condition $-2 - r + t = 0$ corresponds to the H-orbit of row 16 of Table 2, whereas the condition $-4 + r + t = 0$ corresponds to the H-orbit of row 14 of Table 2.

• $\pm(2+2r) \pm r^2 + t = 0$. Observe that, using (iv), the condition $\pm(2+2r+r^2)+t = 0$ is equivalent to $\pm(4+r)+t = 0$, which we know is impossible. Moreover, $-2 - 2r + r^2 + t = 0$ implies $t = 3$, which in view of (vii) implies that $r - 1 = -(r - 1)$, contradicting (ii). Thus we are left with the condition $2 + 2r - r^2 + t = 0$, which is equivalent to $3 + t = 0$. Indeed, such 8-cycles can exist. It is easy to verify that the H-orbit in question corresponds to row 17 of Table 2.

Case 4: out $= 4$. We divide this case into two subcases depending on whether C contains edges from every subgraph $[X_i, X_{i+1}]$ (which can of course occur only when $m = 4$) or not.

Subcase 4.1: C does not contain edges from every $[X_i, X_{i+1}]$. Note that in this case C can have at most three consecutive inner edges. If the maximal number of such edges is 3 or 2, then the H-orbit of C corresponds in a natural way to one of the conditions $3 \pm r^2 = 0$ or $1 \pm 3r^2 = 0$, and $2 \pm 2r^2 = 0$, $2 \pm r \pm r^2 = 0$, $1 \pm 2r \pm r^2 = 0$ or $1 \pm r \pm 2r^2 = 0$, respectively. The 8-cycles for which the maximal number of consecutive inner edges is 1, are dealt with separately.

• $3 \pm r^2 = 0$. If $3 + r^2 = 0$, then (iv) implies that $2 + 2r = 0$, and so $0 = 2r + 2r^2 = -8$, a contradiction. Similarly $3 - r^2 = 0$ cannot hold since in this case $4 - 2r = 0$ which we already know is impossible. In view of Proposition 3.6 the conditions $1 \pm 3r^2 = 0$ are also impossible.

• $2 \pm 2r^2 = 0$, which is equivalent to $2 \pm (4r - 2) = 0$. Clearly $4r = 0$ cannot hold, so we are left with $4 - 4r = 0$. This of course can occur. The corresponding 8-cycles are of code $agzgagzg$ and the length of the H-orbit in question, which clearly corresponds to row 8 of Table 2, is mn, since τ preserves the 8-cycle given as a representative.

• $2 \pm r \pm r^2 = 0$. Observe that (vii) implies that $2 \pm (r - r^2) = 0$ cannot hold, since $2 \notin \langle m \rangle$. If $2 + r + r^2 = 0$, then $1 + 3r = 0$, and so $0 = r + 3r^2 = -3 + 7r$. This forces $5 - r = 0$, which we already know is not possible. Thus we are left with $2 - r - r^2 = 0$, which is equivalent to $3 - 3r = 0$. There are two possible H-orbits of such 8-cycles. The 8-cycles of one of them are of code a^3zgagz, the corresponding H-orbit is of length mn and is given in row 5 of Table 2. The 8-cycles of the other H-orbit are of code az^3agzg, the length of the orbit is $2mn$ and it is given in row 7 of Table 2.

• $1 \pm r \pm 2r^2 = 0$. In view of Proposition 3.6 and the above paragraph we see that the only possibility is $1 + r - 2r^2 = 0$ (which is equivalent to $3 - 3r = 0$). There are two possible H-orbits corresponding to this condition. It is easy to see that the one containing the 8-cycle $u_0^0 u_0^1 u_1^1 u_1^{1+r} u_2^{1+r} u_2^{1+r-r^2} u_2^{1+r-2r^2} u_1^{1+r-2r^2}$ coincides with the H-orbit of row 7 of Table 2 already found in the previous paragraph. The 8-cycles belonging to the other H-orbit, which is of length mn and is given in row 6 of Table 2, are of code a^3zgagz.

• $1 \pm 2r \pm r^2 = 0$. Of course $1 - 2r + r^2 = 0$ always holds by (iv). We have two such H-orbits (recall that C contains two consecutive inner edges in this case), both of length mn. The 8-cycles contained in one of them are of code $azgzazgz$ and the 8-cycles contained in the other are of code $azagazag$. These two H-orbits correspond to rows 2 and 3 of Table 2, respectively. Furthermore, since $r \in \mathbb{Z}_n^*$ and since we already know that $2 - 4r = 0$ cannot hold, none of the other three conditions can hold. For instance, $1 + 2r + r^2 = 0$ would imply $4r = 0$, that is $4 \equiv 0 \pmod{n}$, which is not the case.

• We are now left with the possibility that C does not contain two consecutive inner edges, that is inner and outer edges alternate on C. Depending on whether C contains vertices of two or three orbits X_i we have that the H-orbit of C corresponds in a natural way to a condition of form $1 \pm r \pm 1 \pm r = 0$ or $1 \pm r \pm r^2 \pm r = 0$, respectively. Since we already know that $2 \pm 2r = 0$ cannot hold (recall that $2 + 2r = 0$ implies $8 \equiv 0 \pmod{n}$), the first of these conditions is only possible if we have $1 + r - 1 - r = 0$ (or $1 - r - 1 + r$, which gives the same H-orbit). These 8-cycles always exist, they are of code a^2zaz^2az and the length of the H-orbit in question, which corresponds to row 1 of Table 2, is $2mn$. As for the other set of conditions, (ii) implies that this condition is of form $1 \pm 2r \pm r^2$. By the arguments of the previous paragraph we see that only $1 - 2r + r^2 = 0$ (which always holds) is possible. The corresponding H-orbit is in fact the H-orbit already considered in this paragraph (whose 8-cycles are of code a^2zaz^2az).

Subcase 4.2: C contains edges from every $[X_i, X_{i+1}]$. Of course, $m = 4$ in this case. We divide our analysis depending on the maximal number max of consecutive inner edges of C.

Table 3
The frequencies $f_g(\mathcal{C})$, $f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ in the case when $m \notin \{3, 4, 6\}$

	None	$3(r-1)=0$	$4(r-1)=0$
$f_g(\mathcal{C})$	2	10	4
$f_a(\mathcal{C})$	7	15	8
$f_z(\mathcal{C})$	7	15	8

• max $= 4$. In this case the corresponding condition is of the form $\pm 4 + t = 0$. Combining together (ii), (v) and (vii) we find that $t = 4$ is not possible. On the other hand, $4 + t = 0$ is possible. The H-orbit in question, which corresponds to row 18 of Table 2, is of length $4n$ and its 8-cycles have code g^3zg^3z.

• max $= 3$. In this case the corresponding condition is one of $\pm 3 \pm r + t = 0$, $\pm 3 \pm r^2 + t = 0$ and $\pm 3 \pm r^3 + t = 0$. By (v) and (vii) only $\pm(3 + r) + t = 0$, $\pm(3 + r^2) + t = 0$ and $\pm(3 + r^3) + t = 0$ are possible. If $3 + r + t = 0$, then $u_0^0u_0^1u_0^2u_0^3u_1^3u_1^{3+r}u_2^{3+r}u_3^{3+r}$ is an 8-cycle, and so its image under τ, that is $u_0^0u_1^0u_2^0u_3^0u_3^{r^3}u_0^{r^3+t}u_0^{1+r^3+t}u_0^{2+r^3+t}$, also is an 8-cycle. But then $3+r^3+t = 0$, and so $r^3 - r = 0$, implying that $r^2 = 1$, a contradiction. Note that this also implies that $3+r^3+t = 0$ cannot hold. The remaining four conditions are in fact all possible. The 8-cycles corresponding to $-3 - r + t = 0$ are of code $a^3g^2ag^2$, the H-orbit in question is of length mn and corresponds to row 21 of Table 2. The 8-cycles corresponding to $-3 - r^3 + t = 0$ (which is equivalent to $t = 1 + 3r$) are also of code $a^3g^2ag^2$; the H-orbit in question is of length mn and corresponds to row 23 of Table 2. The 8-cycles corresponding to $-3 - r^2 + t = 0$ (which is equivalent to $t = 2 + 2r$) are of code a^2gag^2ag; the H-orbit in question is of length $2mn$ and corresponds to row 25 of Table 2. Finally, the 8-cycles corresponding to $3 + r^2 + t = 0$ (which is equivalent to $t = -2 - 2r$) are of code g^2zgz^2gz; the H-orbit in question is of length $2mn$ and corresponds to row 27 of Table 2.

• max $= 2$. Note that since C has four inner and four outer edges we can assume (applying τ if necessary), that C also has at most two consecutive outer edges. Thus C corresponds to one of the conditions $\pm 2 \pm 2r^2 + t = 0$, $\pm 2 \pm r \pm r^2 + t = 0$, $\pm 2 \pm r \pm r^3 + t = 0$ and $\pm 2 \pm r^2 \pm r^3 + t = 0$. Observe first that combining together (i), (ii), (iv) and (vii) we find that $n \equiv 0 \pmod{16}$. Thus $\pm(2 - 2r^2) + t = 0$ is not possible, since then $t \equiv 0 \pmod{16}$, which contradicts (v). Moreover, $-2 - 2r^2 + t = 0$ is not possible, since then $-4r + t = 0$, and so $t = 4$, which by (vii) contradicts (ii). The condition $2 + 2r^2 + t = 0$ (which is equivalent to $4 + t = 0$) is possible. The H-orbit in question corresponds to row 19 of Table 2. It is of length $2n$ and its 8-cycles are of code $gzgzgzgz$. As for the remaining conditions, note that combining together (v) and (vii) we find that none of $\pm 2 \pm (r - r^2) + t = 0$, $\pm 2 \pm (r - r^3) + t = 0$ and $\pm 2 \pm (r^2 - r^3) + t = 0$ is possible. Using similar techniques as throughout this proof it is easy to see that the only possibilities (since $n \neq 16$) are $-2 - r - r^2 + t = 0$ (which is equivalent to $-1 - 3r + t = 0$), $2 + r + r^3 + t = 0$ (which is equivalent to $4 + t = 0$) and $-2 - r^2 - r^3 + t = 0$ (which is equivalent to $-3 - r + t = 0$). The corresponding H-orbits are given in rows 24, 20 and 22 of Table 2, respectively. The respective codes of their 8-cycles are a^5gag, gz^3gz^3 and a^5gag.

• max $= 1$. It is clear that in this case inner and outer edges alternate on C. The corresponds condition is thus of the form $\pm 1 \pm r \pm r^2 \pm r^3 + t = 0$. Observe that $1 + r + r^2 + r^3 = 2r + 2r^2 = -2 + 6r = 2 + 2r$. Thus $1 + r + r^2 + r^3 + t = 0$, which is equivalent to $2 + 2r + t = 0$, is possible. The corresponding H-orbit is of length n and its 8-cycles are of code z^8; this orbit is given in row 28 of Table 2. Also $-1 - r - r^2 - r^3 + t = 0$, which is equivalent to $-2 - 2r + t = 0$, is possible. The corresponding H-orbit, given in row 26 of Table 2, is also of length n and its 8-cycles are of code a^8. As for the remaining conditions, they are of form $1 \pm r \pm r^2 \pm r^3 + t = 0$ (with at least one sign being a minus). Combining together (v) and (vii) we find that only $1 - r - r^2 + r^3 + t = 0$ and $1 - r + r^2 - r^3 + t = 0$ could possibly hold. However, none of the two is possible, since multiplication by $r^2 + 1$ or $r + 1$, respectively, gives $2t = 0$, which contradicts (i) and (v).

This completes the proof. ■

5. The proof of Theorem 1.1

Combining together Proposition 3.5 and Lemma 4.1 the proof of our main theorem is now at hand.

Proof of Theorem 1.1. Combining together Propositions 2.2 and 2.4 the forward implication is clear. Suppose then that the parameters m, n, r and t satisfy all the conditions of Theorem 1.1. Letting H be as in Proposition 2.3 we see that H acts half-arc-transitively on $X = \mathcal{Y}(m, n; r, t)$, and so X is either half-arc-transitive or arc-transitive. It is easy to see that the assumptions on the parameters m, n, r and t imply that $n \geq 18$ if n is even. Thus we may apply Lemma 4.1. Combining together the information given in Table 2 and Proposition 3.5 we now show that X cannot be arc-transitive. We remark that we will be using Lemma 3.4 to calculate the frequencies of 2-paths without special reference to it. Let $\mathcal{C}$ denote the set of all 8-cycles of X. The analysis is divided into four cases depending on the parameter m.

Case 1: $m \notin \{3, 4, 6\}$. Observe that $3(r - 1) = 0$ and $4(r - 1) = 0$ cannot both hold. Thus, Table 2 implies that, depending on whether one of $3(r - 1) = 0$ and $4(r - 1) = 0$ holds or not, the frequencies $f_g(\mathcal{C})$, $f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ are as given in Table 3. By Proposition 3.5 X is half-arc-transitive in this case.

Case 2: $m = 6$. Note that $4(r - 1) = 0$ is not possible in this case since then $2(r - 1) = 0$, and so $(r - 1)^2 = 0$ implies that $r^2 = 1$, which does not hold. Letting A_1, A_2, B_1 and B_2 be as in Table 2 we see that the frequencies are as

Table 4
The contributions to frequencies $f_g(\mathcal{C}), f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ of possible nontrivial conditions in the case when $m = 3$

	$t = 4 - r$	$t = 2 + r$	$3 + t = 0$
$f_g(\mathcal{C})$	6	6	2
$f_a(\mathcal{C})$	6	6	2
$f_z(\mathcal{C})$	4	4	4

Table 5
The contributions to frequencies $f_g(\mathcal{C}), f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ of possible nontrivial conditions in the case when $m = 4$

	$4 + t = 0$	$t = 3 + r$	$t = 1 + 3r$	$t = -2 - 2r$
$f_g(\mathcal{C})$	5	3	3	4
$f_a(\mathcal{C})$	0	5	5	0
$f_z(\mathcal{C})$	5	0	0	5

Table 6
The image $u_0^j\varphi$, where $j = 8a + b$, depending on a and b

$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$
u_0^{8a}	u_0^{1+8a}	u_3^{1+8a-t}	$u_3^{1+r^3+8a-t}$	$u_2^{1+r^3+8a-t}$	$u_2^{1+r^2+r^3+8a-t}$	$u_1^{1+r^2+r^3+8a-t}$	$u_1^{8(a+1)}$

given in Table 3 unless (m, n, r, t) belongs to one (or more) of the above four sets. Note that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$. If $(m, n, r, t) = (6, 18, 13, 12) \in A_1 \subset B_1$ (or $(m, n, r, t) = (6, 18, 7, 12) \in A_2 \subset B_2$) then $3(r - 1) = 0$, and so $f_g(\mathcal{C}) = 10 + 8 = 18, f_a(\mathcal{C}) = 15 + 4 = 19$ and $f_z(\mathcal{C}) = 15 + 4 = 19$. If however $(m, n, r, t) = (6, 36, 31, 30) \in B_1 \setminus A_1$ (or $(m, n, r, t) = (6, 36, 7, 30) \in B_2 \setminus A_2$), then $3(r - 1) \neq 0$, and so $f_g(\mathcal{C}) = 2 + 4 = 6, f_a(\mathcal{C}) = 7 + 2 = 9$ and $f_z(\mathcal{C}) = 7 + 2 = 9$. In any case Proposition 3.5 implies that X is half-arc-transitive.

Case 3: $m = 3$. Of course $4(r-1) \neq 0$, and so if none of the H-orbits from rows 9–17 of Table 2 exists, then the frequencies are as in the second column of Table 3. Suppose first that (m, n, r, t) does not belong to $A_1 \cup A_2$. Then the contributions to the frequencies $f_g(\mathcal{C}), f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ of possible conditions, that is $t = 4 - r, t = 2 + r$ and $3 + t = 0$, are given in Table 4. It is now clear that $f_g(\mathcal{C})$ cannot be equal to any one of $f_a(\mathcal{C})$ and $f_z(\mathcal{C})$, implying that X is half-arc-transitive in this case. Suppose then that (m, n, r, t) does belong to $A_1 \cup A_2$. In view of Proposition 3.6 we can assume that $X = \mathcal{Y}(3, 9; 4, 6)$ or $X = \mathcal{Y}(3, 18; 13, 15)$. (Note that this implies that $(m, n, r, t) \in A_1 \subset B_1$.) In both cases $t = 2 + r, 3 + t = 0$ and $t \neq 4 - r$. Thus Table 2 implies that in each of these two cases we have $f_g(\mathcal{C}) = 10+8+8 = 26$ and $f_a(\mathcal{C}) = f_z(\mathcal{C}) = 15+8+4 = 27$, and so X is half-arc-transitive.

Case 4: $m = 4$. Observe that in this case the only possible H-orbits of 8-cycles are those given in rows 1-3, 8 and 18-28. Moreover, the H-orbits of rows 1-3 and 8 always exist. Thus if none of the conditions from rows 18-28 holds, then the frequencies are as in the third column of Table 3. By assumption $t = 2 + 2r$ does not hold. The contribution of each of the possible nontrivial conditions to the frequencies $f_g(\mathcal{C}), f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ is thus as given in Table 5. It is now evident that the only possibility for $f_g(\mathcal{C})$ to equal one of $f_a(\mathcal{C})$ and $f_z(\mathcal{C})$ is when $t = -2-2r$ and $4+t \neq 0, t \neq 3+r, t \neq 1+3r$. However, we now show, that in this case X still cannot be arc-transitive, which then completes the proof.

Suppose on the contrary that X is arc-transitive. As $f_g(\mathcal{C}) = f_a(\mathcal{C}) = 8$ and $f_z(\mathcal{C}) = 13$, Proposition 3.3 implies that the set of zigzags of X is an orbit of Aut$X$, and so Proposition 3.1 implies that any glide can be mapped to an anchor by some automorphism of X. Let $\varphi \in \mathrm{Aut}X$ be an automorphism mapping the glide $u_0^{-1}u_0^0u_0^1$ to one of the two anchors with internal vertex u_0^0. Since Zig$X$ is an orbit of Aut$X$, one of the automorphisms φ and $\tau\varphi$ maps the above glide to the anchor $u_1^0u_0^0u_0^1$; with no loss of generality assume it is φ. In fact, we can assume that $u_0^{-1}\varphi = u_1^0$ (and so $u_1^0\varphi = u_0^{-1}$ since $u_0^{-1}u_0^0u_1^0 \in \mathrm{Zig}X$). Observe that since the only nontrivial condition is $t = -2 - 2r$, the only 8-cycles containing two consecutive glides are the 8-cycles of the H-orbit from row 27 of Table 2. Thus a unique 8-cycle C_1 containing the 3-path $u_0^{-1}u_0^0u_0^1u_0^2$ exists. Note that C_1 is of code g^2zgz^2gz. Combining together the facts that φ maps the glide $u_0^{-1}u_0^0u_0^1$ to an anchor, that Zig$X$ is an Aut$X$-orbit and that the only 8-cycles containing anchors and four zigzags, two of which are consecutive, are the 8-cycles from the H-orbit of row 1 of Table 2, we find that the glide $u_0^0u_0^1u_0^2$ is also mapped to an anchor by φ, that is $u_0^2\varphi = u_3^{1-t}$. Next, the 3-path $u_0^0u_0^1u_0^2u_0^3$ is contained on a unique 8-cycle C_2 (which is also of code g^2zgz^2gz). A similar argument as above shows that $C_2\varphi$ belongs to the H-orbit from row 1 of Table 2, and so $u_0^3\varphi = u_3^{1+r^3-t}$. Note that since $4(r - 1) = t(r - 1) = 0$, we have $1 + r + r^2 + r^3 - t = 2r + 2r^2 - t = 4r + 4r^2 = 4 + 4 = 8$. Thus, continuing inductively we find that the images $u_0^j\varphi$ of the vertices u_0^j are as stated in Table 6, where $a \in \{0, 1, \ldots, \frac{n}{8} - 1\}$ and $b \in \{0, 1, \ldots, 7\}$ are the unique integers for which $j = 8a + b$.

Since $4 + t \neq 0$ and $2t = -4 - 4r = -8$, we find that $t = \frac{n}{2} - 4$. Combining together (ii), (iv) and (vii) we find that $n \equiv 0 \pmod{16}$. Moreover, letting k be as in (vii) we find that $k\frac{n}{2} - 4k = -4k$, and so k is even. Thus $r - 1 \equiv 0 \pmod 8$. Then Table 6 implies that $u_0^{r-1}\varphi = u_0^{r-1}$ and $u_0^r\varphi = u_0^r$. Since $u_0^{r-1}u_0^ru_1^r \in \mathrm{Zig}X$, we must have $u_1^r\varphi = u_1^r$. However, $u_1^0\varphi = u_0^{-1}$, and so the fact that $u_0^0u_1^0u_1^r \in \mathrm{Zig}X$ implies that $u_1^r\varphi = u_3^{-1-t}$, a contradiction. This proves that X is half-arc-transitive, as claimed. ■

This completes the classification of connected quartic half-arc-transitive weak metacirculants of Class II. The natural question which now arises is which of them are pairwise isomorphic. As the next proposition shows the only isomorphic pairs are those given by Proposition 3.6. This enables us to construct all pairwise nonisomorphic connected quartic half-arc-transitive weak metacirculants of Class II up to a given order quite easily. For instance, it turns out that there are only three such graphs up to order 100, these are the graphs $\mathcal{Y}(3, 9; 4, 6)$, $\mathcal{Y}(3, 18; 7, 15)$ and $\mathcal{Y}(3, 27; 10, 24)$. Furthermore, there are 198 such graphs up to order 1000 and there are 3915 such graphs up to order 10 000.

Proposition 5.1. *Let $X = \mathcal{Y}(m, n; r, t)$ and $Y = \mathcal{Y}(m', n'; r', t')$, where the parameters m, n, r and t, as well as m', n', r' and t', satisfy the conditions* (i)–(viii) *of Theorem 1.1. Then the graphs X and Y are isomorphic if and only if $m' = m$, $n' = n$, $t' = t$ and we either have that $r' = r$ or $r' = r^{-1}$.*

Proof. By Theorem 1.1 the graphs X and Y are both half-arc-transitive. It follows from [32, Theorem 5.1] that vertex-stabilizers in $\mathrm{Aut}X$ and $\mathrm{Aut}Y$ are isomorphic to $\mathbb{Z}_2$. In view of Proposition 3.6 we only need to show the forward implication. Denote the vertices of X by u_i^j, $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$, and the vertices of Y by v_i^j, $i \in \mathbb{Z}_{m'}, j \in \mathbb{Z}_{n'}$, with edges as usual. Suppose then that $X \cong Y$ and let $\varphi : X \to Y$ be an isomorphism mapping u_0^0 to v_0^0 and mapping the inner edge $u_0^0u_0^1$ of X to an inner edge of Y, that is, we either have $u_0^1\varphi = v_0^1$ or $u_0^1\varphi = v_0^{-1}$.

We first show that φ maps the anchors of X to the anchors of Y and that either $(\mathrm{Gli}X)\varphi = \mathrm{Zig}Y$ and $(\mathrm{Zig}X)\varphi = \mathrm{Gli}Y$, or $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$ and $(\mathrm{Zig}X)\varphi = \mathrm{Zig}Y$ holds. Suppose first that φ maps a nonanchor of X to an anchor of Y. Then letting τ' be the unique nontrivial element of $(\mathrm{Aut}Y)_v$, where v is the internal vertex of this anchor, we see that $\varphi\tau'\varphi^{-1} \in \mathrm{Aut}X$ flips the above nonanchor of X, which clearly contradicts the fact that X is half-arc-transitive. It is easy to see that in view of the fact that $(\mathrm{Aut}X)_u \cong \mathbb{Z}_2$ for all vertices u of X implies that if φ maps some glide of X to a glide, respectively zigzag, of Y, then $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$, respectively $(\mathrm{Gli}X)\varphi = \mathrm{Zig}Y$ holds, which proves our claim. We now distinguish two cases.

Suppose first that $(\mathrm{Gli}X)\varphi = \mathrm{Zig}Y$ and $(\mathrm{Zig}X)\varphi = \mathrm{Gli}Y$ holds. By Lemma 4.1 the only 8-cycles of code a^2gag^2ag are those belonging to the H-orbit from row 25 of Table 2. Since the 8-cycles of the H-orbit from row 1 of that table (which always exist) have code a^2zaz^2az it follows that the H-orbit from row 25 of Table 2 exists in X as well as in Y. Hence $m = m' = 4$, implying that $n = n'$, and $t = 2 + 2r$, $t' = 2 + 2r'$ hold. By (iv) we have that $4(r - 1) = 4(r' - 1) = 0$, and so (ii) and (iv) imply that $2(r - 1) = 2(r' - 1) = \frac{n}{2}$. Hence, $t = t'$. Finally, since we now have $2(r - r') = 0$, we either have that $r = r'$ or $r' = \frac{n}{2} + r$. But as $r(\frac{n}{2} + r) = \frac{n}{2} + 2r - 1 = 1$, we have that $r' = r^{-1}$ in this case.

Suppose now that $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$ and $(\mathrm{Zig}X)\varphi = \mathrm{Zig}Y$ holds. Since φ maps the inner edge $u_0^0u_0^1$ to an inner edge of Y it is now clear that $X_1\varphi = Y_1 = \{v_0^j \mid j \in \mathbb{Z}_{n'}\}$, implying that $n = n'$, and hence also $m = m'$. Let us first consider the possibility that $u_0^1\varphi = v_0^{-1}$. In this case $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$ implies that $u_0^t\varphi = v_0^{-t}$. On the other hand, as $u_0^1u_0^0u_1^0 \in \mathrm{Anc}^-X$, we have that $u_1^0\varphi = v_{m-1}^{-t'}$, and so $(\mathrm{Gli}X)\varphi = \mathrm{Gli}Y$ implies that also $u_0^t\varphi = v_0^{-t'}$ holds. We thus have $t' = t$. Since $u_0^{r-1}u_0^ru_1^r \in \mathrm{Zig}X$, we must have $u_1^r\varphi = v_{m-1}^{-r-t}$. On the other hand, $u_0^0u_1^0u_1^r \in \mathrm{Zig}X$ implies that $u_1^r\varphi = v_{m-1}^{-r'^{m-1}-t}$. Hence $-r = -r'^{m-1}$, that is, $r' = r^{-1}$. That $t' = t$ and $r' = r$ holds in the case when $u_0^1\varphi = v_0^1$ is straightforward and is left to the reader. ■

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